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Journal of Nonlinear and Convex Analysis

Yokohama Publishers

Journal of Nonlinear and Convex Analysis
Volume 22, Number 10, 2021, 2231–2245

MULTI-GRANULATION INTERVAL-VALUED FUZZY ROUGH SET MODEL UNDER HESITANT ENVIRONMENT

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Abstract. With the swift development of the information era, we have to deal with a mass of data, accompanied by information missing, information ambiguity and other varieties of problems. As two effective theories to cope with the uncertainty, both rough set (RS) and fuzzy set (FS) are extensively employed in various processes of decision. Nevertheless, when the dimension and scale of a data set are greatly enormous, the process of inference and decision-making has become so complex that it has led to hesitation and vacillation of people between several options. In order to surmount the aforesaid limitations, a novel and innovative model referred as multi-granulation interval-valued hesitant fuzzy rough set (MGRS-IVHFS) is established on the foundation of interval-valued hesitant fuzzy rough set (IVHFRS). We recommend the fundamental definitions and relevant properties concerning this model under the circumstance of optimism and pessimism, and calculate the upper and lower approximations in two situations by examples as well as prove the availability and validity of pertinent theorems.

1. Introduction

As the era of big data is racing ahead, intricate information can come into sight everywhere in daily life. In the face of such complex big data, it is potential to generate some situations such as information ambiguity and uncertainty. Designed to solve these uncertainties, rough set (RS) theory was proposed. Especially in artificial intelligence, RS has a vast range of utilizations. RS theory, brought up by Pawlak [9,10] in 1982, is a generalization of classical set. The prime feature of RS is that in the light of classification, the unknown concept is depicted by defining two approximate operators. It links knowledge with classification, and considers knowledge as the ability of classification. The inaccurate or uncertain knowledge can be approached through known knowledge. In recent years, this theory has led to a heated discussion for an increasing number of researchers [3,6,7].

The fuzzy set (FS) pays attention to fuzziness when dealing with problems proposed by Zadeh [24], which generalizes the classical set theory. After the FS were proposed, several related extension forms were also introduced. In 1975, Dubois and Prade [4] offered a proposal about the interval-valued fuzzy set (IVFS), and extended the membership of FS to a subinterval of the interval $[0, 1]$. At the outset of the 21st century, with a view to sequencing diverse interval values, Xu and Da [17] offered a solution by providing two equations to estimate the magnitude between

2020 *Mathematics Subject Classification.* 03E72, 68T30, 68U35.

Key words and phrases. Hesitant fuzzy set, interval-valued, multi-granulation, rough set.

*Corresponding author. This work is supported by the National Natural Science Foundation of China (No.61976245, 61772002).

two interval values, thereafter they structured a matrix whose rows were corresponding to respective interval values for acquiring final verdict. In 1986, Atanassov [1] generalized the FS, and the thought of intuitionistic fuzzy set (IFS) was proposed accordingly. It further described the uncertainty by adding non-membership function. However, with the vigorous advance of the age, issues that people confront with are becoming complex, at the same time the occurrence of uncertainties is augmenting in the decision-making process. Therefore, experts often hesitate and vacillate between several values when evaluation. As a result, Torra [16] brought forward the viewpoint of hesitant fuzzy set (HFS), so that the membership of an object can be composed of a set containing multiple possible values. In 2013, Chen et al. [2] proposed an interval-valued hesitant fuzzy set (IVHFS) based on the HFS.

In addition, granular computing [18,23,25] is also a novel theory, which combines the research achievement of RS, FS, artificial intelligence and other theories. The classical and generalized rough set were almost all set up under single-granulation relation. In granular computing, it is widely shared that an equivalence relation over a domain is regarded as a granularity, plus a partition is regarded as a granularity space. When a given domain is induced by multiple equivalence relations, multi-granulation spaces will be formed. Besides, the resulting RS is known as multi-granulation rough set (MGRS). The construct of MGRS was brought forward by Qian et al. [11,12] in 2010. In 2012, Xu et al. [20] explored MGRS in ordered information systems and proposed two novel models. Subsequently, Xu et al. [21] came up with another model under the name of multi-granulation fuzzy rough set (MGFRS). They have accomplished some research on pertinent properties.

In real life, since the equivalence relation of Pawlak RS is extremely strict, a part of problems cannot be explained by RS model. Consequently, it is crucial to generalize and extend it. For the purpose of overcoming these limitations of classical RS, the research of combining RS and FS [8,19] has become a emerging topic. In 1990, Dubois and Prade [5] presented a concept of fuzzy rough set (FRS), which made the RS reach a momentous development in the fuzzy theory. According to the fusion of HFS and RS, Yang et al. [22] developed axiomatic approaches in connection with hesitant fuzzy rough set (HFRS). In 2014, Zhang et al. [27] combined the IVHFS and RS based on the HFS, and made a point about the interval-valued hesitant fuzzy rough set (IVHFRS), which represented all possible membership degrees of an object by multiple interval values. Based on the model of the IVHFRS over two domains, Zhang et al. [26] proposed a way to diagnose steam turbine fault.

In the information age, we are often at a loss when it comes to reasoning and making decisions. As one of plentiful approaches of knowledge discovery, the strength of RS is not concerned about those additional information and preparation. By virtue of unexpected obstacles and interferences with issues such as missing and repetition during data collection in daily routine, what makes us tough is to characterize the degree of membership by a real number. Depending on the complexity of situations, people tend to select no less than one value to delegate their own standpoints, which guarantees the decision outcome more accurate. moreover, it is thinking about problems from different points of view that makes us find the most appropriate decision-making approach. Thereby, we commence studying the multi-granulation interval-valued hesitant fuzzy rough set (MGRS-IVHFS), that is more objective than other conventional methods. Recently, RS in association with other models is applied by some scholars in terms of prediction [15], multi-criteria decision making [13] and distance measurement [14].

In this paper, according to the properties of IVHFRS, multi-granulation interval-valued fuzzy rough set model under hesitant environment is put forward, and the model in optimistic and pessimistic cases are studied. Moreover, relevant theorems and proofs are given. The arrangement of remanent paper is as following statements. In the first place, the IVHFS is briefly reviewed in part 2. Additionally, the basic foundation of MGRS is proposed. In part 3, the prime notion of the MGRS-IVHFS is taken on. Afterwards, correlative properties of the model and its proof are given. Then the approximations of this model are calculated with instances to certify the validity of theorems. In part 4, we draw a conclusion about this paper and point out the orientation of prospective work.

2. Preliminaries

About this section, some foundational knowledge and properties with reference to the IVHFS will be retrospected firstly. Furthermore, we are going to introduce the primary definitions of MGRS.

For abiding by the principle of conciseness and explicitness, we make an assumption that $\mho$ is a finite and non-empty universe of discourse. Moreover, we abbreviate the lower and upper approximations as the $L\&U$ approximations.

2.1. **Interval-valued hesitant fuzzy set.** Initially, Chen et al. [2] presented an idea about the IVHFS, which was based on the HFS and generated by replacing the clear values in the FS with the interval numbers. It is a more flexible structure that reflects the hesitant degrees of experts when evaluating objects or selecting targets. Prior to introducing the IVHFS, we are about to display the idea of HFS.

Definition 2.1 (See [16]). Assume $\mho$ be a universe with finite elements, then we express a HFS ξ over $\mho$ as $\xi = \{\langle \omega, h_\xi(\omega)\rangle | \omega \in \mho\}$, where $h_\xi(\omega)$ is a set composed by several disparate and finite elements in $[0, 1]$, indicating all possible membership degrees of ω in $\mho$ to set ξ, and the denotation of $h_\xi(\omega)$ is hesitant fuzzy element (HFE).

Definition 2.2 (See [2]). Assume $\mho$ be a universe with finite elements, then we express an IVHFS I over $\mho$ as $I = \{\langle \omega, h_I(\omega)\rangle | \omega \in \mho\}$, where $h_I(\omega)$ is a set composed by several disparate and finite interval numbers in $[0, 1]$, indicating all possible interval-valued membership degrees of ω in $\mho$ to set I, and the denotation of $h_I(\omega)$ is interval-valued hesitant fuzzy element (IVHFE). We can express $h_I(\omega)$ consisting of n interval numbers as $h_I(\omega) = \{\nu_i | i = 1, 2, \ldots, n\}$, where the interval number is $\nu_i = [{\nu_i}^L, {\nu_i}^U]$. Here ${\nu_i}^L$, ${\nu_i}^U$ are the lower and upper limits of the interval.

In quick succession, we recommend two particular interval-valued hesitant fuzzy sets (IVHFSs):

(1) For any $\omega \in \mho, h_I(\omega) = \{[0, 0]\} \Leftrightarrow I$ is denoted as an empty IVHFS. That is to say, the membership degrees of all ω in $\mho$ to set I is 0.

(2) For any $\omega \in \mho, h_I(\omega) = \{[1,1]\} \Leftrightarrow I$ is denoted as a full IVHFS. That is to say, the membership degrees of all ω in $\mho$ to set I is 1.

As can be observed, the number of interval values in diverse IVHFEs is perhaps diverse. Consequently, the following assumption is made:

Assume $h_U(\omega)$ and $h_V(\omega)$ be two IVHFEs. If the lengths of $h_U(\omega)$ and $h_V(\omega)$ are inequable, namely $l(h_U(\omega)) \neq l(h_V(\omega))$, then with the purpose of operating between two IVHFEs, the lengths of $h_U(\omega)$ and $h_V(\omega)$ should be equal as well as both of them are $l = \max\{l(h_U(\omega)), l(h_V(\omega))\}$. If $l(h_U(\omega)) < l(h_V(\omega))$, then we are required to extend $h_U(\omega)$. In other words, we should add its maximum interval number to $h_U(\omega)$ until $l(h_U(\omega)) = l(h_V(\omega))$.

Definition 2.3 (See [27]). Assume the interval-valued hesitant fuzzy relation $\Re$ be an IVHF subset of $\mho \times \mho$ and $\mho$ be a universe with finite elements. We denote $\Re$ by $\Re = \{\langle(\omega,\psi), h_\Re(\omega,\psi)\rangle | (\omega,\psi) \in \mho \times \mho\}$, where $h_\Re(\omega,\psi) : \mho \times \mho \to d[0,1]$. $h_\Re(\omega,\psi)$ is a set of interval numbers in $d[0,1]$, representing all possible membership degrees between ω and ψ.

For the sake of convenience, the family of interval-valued hesitant fuzzy relations over $\mho$ is denoted by $\mathbb{IVHFR}(\mho \times \mho)$. We will recommend the definitions of upper and lower approximations of interval-valued hesitant fuzzy rough set (IVHFRS) in what follows.

Definition 2.4 (See [27]). Assume $\Re$ be a relation over a universe of discourse $\mho$ and $\Re \in \mathbb{IVHFR}(\mho \times \mho)$, then $(\mho, \Re)$ is referred as an approximation space of IVHF. Given any $G \in IVHFS$, then we define the $L\&U$ approximations of G with respect to $(\mho, \Re)$ as $\underline{\Re}(G)$ and $\overline{\Re}(G)$, denoted by

$$\underline{\Re}(G) = \{\langle \omega, h_{\underline{\Re}(G)}(\omega)\rangle | \omega \in \mho\},\ \overline{\Re}(G) = \{\langle \omega, h_{\overline{\Re}(G)}(\omega)\rangle | \omega \in \mho\},$$

where $h_{\underline{\Re}(G)}(\omega) = \wedge_{\psi\in\mho}\{h_{\Re^c}(\omega,\psi) \vee h_G(\psi)\}, h_{\overline{\Re}(G)}(\omega) = \vee_{\psi\in\mho}\{h_\Re(\omega,\psi) \wedge h_G(\psi)\}$. "$\wedge$" signifies "*select smaller*" and "$\vee$" signifies "*select larger*". $(\underline{\Re}(G), \overline{\Re}(G))$ is referred as the IVHFRS of G to $(\mho, \Re)$.

2.2. **Multi-granulation rough set.** It is well acknowledged that when a given domain is induced by multiple relations, divisions can be regarded as multiple granularities, which determine the corresponding multi-granulation rough set (MGRS).

Definition 2.5 (See [12]). Assume triple tuple $\mathcal{I} = (\mho, AT, F)$ be an integrated information system, $A_1,\ A_2, \ldots,\ A_s \in AT$, $\Re_i (i = 1, 2, \ldots, s)$ is the corresponding relation of the attributes in AT. For any $K \subseteq \mho$, the optimistic multi-granulation $L\&U$ approximations of K based on the relation $\Re_i$ are described as $\underline{apr}^O_{\sum_{i=1}^{s} A_i}(K)$ and $\overline{apr}^O_{\sum_{i=1}^{s} A_i}(K)$, denoted by

$$\underline{apr}^O_{\sum_{i=1}^{s} A_i}(K) = \{\omega | \bigvee_{i=1}^{s}([\omega]_{A_i} \subseteq K)\},\ \overline{apr}^O_{\sum_{i=1}^{s} A_i}(K) = \{\omega | \bigwedge_{i=1}^{s}([\omega]_{A_i} \cap K \neq \emptyset)\}$$

where "$\vee$" signifies "*or*" and "$\wedge$" signifies "*and*".

Additionally, if $\underline{apr}^{O}_{\sum_{i=1}^{s} A_i}(K) = \overline{apr}^{O}_{\sum_{i=1}^{s} A_i}(K)$, then K is referred to be an optimistic multi-granulation definable set with respect to multi-granulation structures $A_1, A_2, \ldots, A_s$. Otherwise, K is an optimistic multi-granulation rough set in an information system.

Analogously, for any $K \subseteq \mho$, the pessimistic multi-granulation $L\&U$ approximations of K based on the relation $\Re_i$ are described as

$$\underline{apr}^{P}_{\sum_{i=1}^{s} A_i}(K) = \{\omega | \bigwedge_{i=1}^{s}([\omega]_{A_i} \subseteq K)\},\ \overline{apr}^{P}_{\sum_{i=1}^{s} A_i}(K) = \{\omega | \bigvee_{i=1}^{s}([\omega]_{A_i} \cap K \neq \emptyset)\},$$

where "$\vee$" signifies "*or*" and "$\wedge$" signifies "*and*".

3. Multi-granulation interval-valued hesitant fuzzy rough set

The previous section is an introduction to some essential notions and relevant properties of IVHFS. As for the next narrative, we will further extend the IVHFS to multi-granulation spaces.

3.1. The optimistic multi-granulation interval-valued hesitant fuzzy rough set. For the imminent contents, the MGRS-IVHFS that we put forward has two forms, including the optimistic and pessimistic multi-granulation interval-valued hesitant fuzzy rough sets (MGRS-IVHFSs). Here, we will take into account the former induced by multiple interval-valued hesitant fuzzy relations.

Definition 3.1. Assume $\Re_i\,(i = 1, 2, \cdots, s)$ be s relations over a universe of discourse $\mho$, and $\Re_i \in \mathbb{IVHFR}(\mho \times \mho)$. $(\mho, \Re_i)$ is termed as an approximation space of MGRS-IVHFS. Given any $G \in IVHFS$, then we define the optimistic MGRS-IVHFS $L\&U$ approximations of G with respect to $(\mho, \Re_i)$ as $\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)$ and $\overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)$ as below:

$$!\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G) = \{\langle \omega, h_{\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)}(\omega)\rangle | \omega \in \mho\},$$

$$\overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G) = \{\langle \omega, h_{\overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)}(\omega)\rangle | \omega \in \mho\},$$

where $h_{\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)}(\omega) = \bigvee_{i=1}^{s} \wedge_{\psi \in \mho}\{h_{\Re_i^c}(\omega, \psi) \vee h_G(\psi)\}$, $h_{\overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)}(\omega) = \bigwedge_{i=1}^{s} \vee_{\psi \in \mho}$ $\{h_{\Re_i}(\omega, \psi) \wedge h_G(\psi)\}$. "$\wedge$" signifies "*select smaller*" and "$\vee$" signifies "*select larger*". $(\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G), \overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G))$ is termed as the optimistic MGRS-IVHFS of G to $(\mho, \Re_i)$.

Additionally, if $\underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G) = \overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G)$, then an IVHFS G is optimistic and definable under multi-granulation relations. Otherwise, G is optimistic and rough.

Proposition 3.2. *Suppose that $(\mho, \Re_i)$ is an approximation space of MGRS-IVHFS. Given $G, K \in IVHFS$ over $\mho$, then the optimistic MGRS-IVHFS satisfies the following properties:*

$(OL_1)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \subseteq G,$ $(OU_1)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \supseteq G,$

$(OL_2)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G^c) = (\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G))^c,$ $(OU_2)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G^c) = (\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G))^c,$

$(OL_3)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(\emptyset) = \emptyset,$ $(OU_3)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(\emptyset) = \emptyset,$

$(OL_4)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(\mho) = \mho,$ $(OU_4)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(\mho) = \mho,$

$(OL_5)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G \cap K) = \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \cap \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(K),$

$(OU_5)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G \cup K) = \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \cup \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(K),$

$(OL_6)G \subseteq K \Rightarrow \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(K),$

$(OU_6)G \subseteq K \Rightarrow \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(K),$

$(OL_7)\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G \cup K) \supseteq \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \cup \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(K),$

$(OU_7)\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G \cap K) \subseteq \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \cap \overline{M}^O_{\sum_{i=1}^{s}\Re_i}(K).$

Proof. Without loss of generality, we merely demonstrate the properties of lower approximation, then it is straightforward to obtain these properties of upper approximation in an analogous technique.

(OL_1) for every $\omega \in \underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)$, it can be obtained from Definition 3.1 that for every $\omega, \psi \in \mho$, $\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) = \{\langle\omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi)\}\rangle|\omega \in \mho\} = \{\langle\omega, \overset{s}{\underset{i=1}{\vee}}\{\wedge_{\psi\in\mho}\{(h_{\Re_i^c}(\omega,\omega)\vee h_G(\omega))\}\wedge\{\underset{\psi\neq\omega}{\wedge}(h_{\Re_i^c}(\omega,\psi)\vee h_G(\psi))\}\}\rangle\} = \{\langle\omega, \overset{s}{\underset{i=1}{\vee}}\{\wedge_{\psi\in\mho}\{\{[0,0]\} \vee h_G(\omega)\} \wedge \{\underset{\psi\neq\omega}{\wedge}(h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi))\}\}\rangle\} = \{\langle\omega, \overset{s}{\underset{i=1}{\vee}}\{\wedge_{\psi\in\mho}\{h_G(\omega) \wedge \{\underset{\psi\neq\omega}{\wedge}(h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi))\}\}\}\rangle\}$. So we have $h_G(\omega) \wedge \{\underset{\psi\neq\omega}{\wedge}(h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi))\}\preceq h_G(\omega)$, thus $\overset{s}{\underset{i=1}{\vee}}\{h_G(\omega) \wedge \{\underset{\psi\neq\omega}{\wedge}(h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi))\}\}\preceq h_G(\omega)$. It is clear to figure out $h_{\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)\preceq h_G(\omega)$, thereby $\underline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) \subseteq G$ is proved.

(OL_2) $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G^c) = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in \mho}\{h_{\Re_i^c}(\omega,\psi) \vee h_{G^c}(\psi)\}\rangle | \omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in \mho}$ $\{(\sim h_{\Re_i}(\omega,\psi)) \vee (\sim h_G(\psi))\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{\sim (h_{\Re_i}(\omega,\psi) \wedge h_G(\psi))\}\rangle|\omega \in$ $\mho\} = \{\langle \omega, \sim \{\overset{s}{\underset{i=1}{\wedge}} \vee_{\psi\in\mho}(h_{\Re_i}(\omega,\psi) \wedge h_G(\psi))\}\rangle|\omega \in \mho\} = (\overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G))^c$, so we get $\overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G^c) = (\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G))^c$.

(OL_3) For every $\omega \in \mho$, if $G = \emptyset$, then $h_G(\psi) = \{[0,0]\}$. It can be found that $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(\emptyset) = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee \{[0,0]\}\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho} h_{\Re_i^c}(\omega,\psi)\rangle|$ $\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}}\{h_{\Re_i^c}(\omega,\omega) \wedge (\underset{\psi\neq\omega}{\wedge} h_{\Re_i^c}(\omega,\psi))\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}}\{\{[0,0]\} \wedge$ $(\underset{\psi\neq\omega}{\wedge} h_{\Re_i^c}(\omega,\psi))\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}}\{[0,0]\}\rangle|\omega \in \mho\} = \{\langle \omega, \{[0,0]\}\rangle|\omega \in \mho\}$, therefore $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(\emptyset) = \emptyset$.

(OL_4) For every $\omega \in \mho$, if $G = \mho$, then $h_G(\psi) = \{[1,1]\}$. It can be found that $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(\mho) = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee \{[1,1]\}\}\rangle|\omega \in \mho\} = \{\langle \omega, \{[1,1]\}\rangle|\omega \in \mho\}$, therefore $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(\mho) = \mho$.

(OL_5) For every $\omega \in \mho$, it is distinct that $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G\cap K) = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi)$ $\vee\, h_{G\cap K}(\psi)\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee (h_G(\psi) \wedge h_K(\psi))\}\rangle|\omega \in \mho\} =$ $\{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{(h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi)) \wedge (h_{\Re_i^c}(\omega,\psi) \vee h_K(\psi))\}\rangle|\omega \in \mho\} = \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}$ $\{h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi)\}\rangle|\omega \in \mho\} \wedge \{\langle \omega, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee h_K(\psi)\}\rangle|\omega \in \mho\}$
$= h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\omega) \wedge h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)}(\omega) = \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \cap \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)$.

(OL_6) Because of $G \subseteq K$, we learn about $h_G(\psi) \preceq h_K(\psi)$. That is to say, $h_G^{(k)L}(\psi) \leq h_K^{(k)L}(\psi), h_G^{(k)U}(\psi) \leq h_K^{(k)U}(\psi)$. Thus, $h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\omega) = [\overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}$ $\{h^{(k)L}_{\Re_i^c}(\omega,\psi) \vee h_G^{(k)L}(\psi)\}, \overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h^{(k)U}_{\Re_i^c}\omega,\psi) \vee h_G^{(k)U}(\psi)\}] \leq [\overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h^{(k)L}_{\Re_i^c}(\omega,\psi) \vee$ $h_K^{(k)L}(\psi)\}$,
$\overset{s}{\underset{i=1}{\vee}} \wedge_{\psi\in\mho}\{h^{(k)U}_{\Re_i^c}(\omega,\psi) \vee h_K^{(k)U}(\psi)\}] = h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)}(\omega)$. So we get $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq$ $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)$. $G \subseteq K \Rightarrow \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)$ is proved.

(OL_7) By reason of $G \subseteq G \cup K$ and $K \subseteq G \cup K$, from this property of (OL_6), it satisfies $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G \cup K)$ and $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G \cup K)$, we can obtain $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G \cup K) \supseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \cup \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(K)$. □

Example 3.3. Table 1, 2 and 3 depict all possible membership degrees between τ_i and $\tau_j (i, j = 1, 2, 3, 4)$, associated with some indicators impacting on the life span of abalone. Given a domain $\mho = \{\tau_1, \tau_2, \tau_3, \tau_4\}$. They are respectively representative of diameter, height, length and weight. $\Re_i \in \mathbb{IVHFR}(\mho \times \mho)(i = 1, 2, 3)$ are three relations over $\mho$ on behalf of proposals from three specialists. Assume $G \in IVHFS$ be a sample of abalone that we have fetched as follows:

$$\{\langle \tau_1, \{[0, 0.3]\}\rangle, \langle \tau_2, \{[0.2, 0.5], [0.3, 0.6]\}\rangle, \langle \tau_3, \{[0.6, 0.8]\}\rangle, \langle \tau_4, \{[0.1, 0.3], [0.4, 0.9]\}\rangle\}.$$

Table 1. An interval-valued hesitant fuzzy relation under $\Re_1$

$\Re_1$	τ_1	τ_2	τ_3	τ_4
τ_1	[1, 1]	[0, 0]	0.2, 0.3], [0.4, 0.5]	[0.1, 0.2], [0.3, 0.5]
τ_2	[0, 0]	[1, 1]	[0.5, 0.6], [0.5, 0.7]	[0.8, 0.9]
τ_3	[0.2, 0.3], [0.4, 0.5]	[0.5, 0.6], [0.5, 0.7]	[1, 1]	[0.3, 0.6], [0.4, 0.7]
τ_4	[0.1, 0.2], [0.3, 0.5]	[0.8, 0.9]	[0.3, 0.6], [0.4, 0.7]	[1, 1]

Table 2. An interval-valued hesitant fuzzy relation under $\Re_2$

$\Re_2$	τ_1	τ_2	τ_3	τ_4
τ_1	[1, 1]	[0.6, 0.7], [0.6, 0.8]	[0.5, 0.7], [0.6, 0.9]	[0.6, 0.7]
τ_2	[0.6, 0.7], [0.6, 0.8]	[1, 1]	[0.2, 0.4]	[0, 0]
τ_3	[0.5, 0.7], [0.6, 0.9]	[0.2, 0.4]	[1, 1]	[0.3, 0.5], [0.6, 0.8]
τ_4	[0.6, 0.7]	[0, 0]	[0.3, 0.5], [0.6, 0.8]	[1, 1]

Table 3. An interval-valued hesitant fuzzy relation under $\Re_3$

$\Re_3$	τ_1	τ_2	τ_3	τ_4
τ_1	[1, 1]	[0.6, 0.8], [0.8, 0.9]	[0.2, 0.3]	[0.5, 0.6], [0.7, 0.9]
τ_2	[0.6, 0.8], [0.8, 0.9]	[1, 1]	[0.2, 0.3], [0.2, 0.5]	[0.1, 0.3], [0.2, 0.6]
τ_3	[0.2, 0.3]	[0.2, 0.3], [0.2, 0.5]	[1, 1]	[0, 0]
τ_4	[0.5, 0.6], [0.7, 0.9]	[0.1, 0.3], [0.2, 0.6]	[0, 0]	[1, 1]

According to the descriptions of Table 1, 2 and 3, we can calculate that

$$h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_1) = \{[0, 0.3]\},\ h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_2) = \{[0.2, 0.4], [0.3, 0.6]\},$$

$$h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_3) = \{[0.6, 0.8], [0.5, 0.8]\},\ h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_4) = \{[0.1, 0.3], [0.3, 0.6]\}.$$

And, we can get a conclusion that the optimistic MGRS-IVHFS lower approximation of G is $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) = \{\langle \tau_1, \{[0, 0.3]\}\rangle, \langle \tau_2, \{[0.2, 0.4], [0.3, 0.6]\}\rangle, \langle \tau_3,$ $\{[0.6, 0.8], [0.5, 0.8]\}\rangle, \langle \tau_4, \{[0.1, 0.3], [0.3, 0.6]\}\rangle | \tau \in \mho\}$.

Analogously, we can also compute that

$$h_{\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_1)=\{[0.2,0.3],[0.4,0.5]\},\ h_{\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_2)=\{[0.2,0.5],[0.3,0.6]\},$$
$$h_{\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_3)=\{[0.6,0.8]\},\ h_{\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G)}(\tau_4)=\{[0.1,0.3],[0.4,0.9]\}.$$

So, it is not difficult to obtain that the optimistic MGRS-IVHFS upper approximation of G is $\overline{M}^O_{\sum_{i=1}^{s}\Re_i}(G) = \{\langle\tau_1,\{[0.2,0.3],[0.4,0.5]\}\rangle,\langle\tau_2,\{[0.2,0.5],[0.3,0.6]\}\rangle,\langle\tau_3,\{[0.6,0.8]\}\rangle,\langle\tau_4,[0.1,0.3],[0.4,0.9]\}\rangle|\tau\in\mho\}$.

3.2. **The pessimistic multi-granulation interval-valued hesitant fuzzy rough set.** In a similar way, next we will consider another model, which is the pessimistic MGRS-IVHFS.

Definition 3.4. Assume $\Re_i\,(i=1,2,\ldots,s)$ be s relations over a universe of discourse $\mho$, and $\Re_i\in\mathbb{IVHFR}(\mho\times\mho)$. $(\mho,\Re_i)$ is termed as an approximation space of MGRS-IVHFS. Given any $G\in IVHFS$, then we define the pessimistic MGRS-IVHFS $L\&U$ approximations of G with respect to $(\mho,\Re_i)$ as $\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)$ and $\overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)$ as below:

$$\begin{aligned}\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)=\{\langle\omega,h_{\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)\rangle|\omega\in\mho\},\ \overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)\\=\{\langle\omega,h_{\overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)\rangle|\omega\in\mho\},\end{aligned}$$

where $h_{\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)=\bigwedge_{i=1}^{s}\wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi)\vee h_G(\psi)\}$, $h_{\overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)=\bigvee_{i=1}^{s}\vee_{\psi\in\mho}\{h_{\Re_i}(\omega,\psi)\wedge h_G(\psi)\}$. "$\wedge$" signifies "*select smaller*" and "$\vee$" signifies "*select larger*". $(\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G),\overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G))$ is termed as the pessimistic MGRS-IVHFS of G to $(\mho,\Re_i)$.

Additionally, if $\underline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)=\overline{M}^P_{\sum_{i=1}^{s}\Re_i}(G)$, then an IVHFS G is pessimistic and definable under multi-granulation relations. Otherwise, G is pessimistic and rough.

Proposition 3.5. *Suppose that* $(\mho, \Re_i)$ *is an approximation space of MGRS-IVHFS. Given* $G, K \in IVHFS$ *over* $\mho$*, then the pessimistic MGRS-IVHFS satisfies the following properties:*

$(PL_1)\underline{M}^P_{\sum_{i=1}^s \Re_i}(G) \subseteq G,$ $(PU_1)\overline{M}^P_{\sum_{i=1}^s \Re_i}(G) \supseteq G,$

$(PL_2)\underline{M}^P_{\sum_{i=1}^s \Re_i}(G^c) = (\overline{M}^P_{\sum_{i=1}^s \Re_i}(G))^c,$ $(PU_2)\overline{M}^P_{\sum_{i=1}^s \Re_i}(G^c) = (\underline{M}^P_{\sum_{i=1}^s \Re_i}(G))^c,$

$(PL_3)\underline{M}^P_{\sum_{i=1}^s \Re_i}(\emptyset) = \emptyset,$ $(PU_3)\overline{M}^P_{\sum_{i=1}^s \Re_i}(\emptyset) = \emptyset,$

$(PL_4)\underline{M}^P_{\sum_{i=1}^s \Re_i}(\mho) = \mho,$ $(PU_4)\overline{M}^P_{\sum_{i=1}^s \Re_i}(\mho) = \mho,$

$(PL_5)\underline{M}^P_{\sum_{i=1}^s \Re_i}(G \cap K) = \underline{M}^P_{\sum_{i=1}^s \Re_i}(G) \cap \underline{M}^P_{\sum_{i=1}^s \Re_i}(K),$

$(PU_5)\overline{M}^P_{\sum_{i=1}^s \Re_i}(G \cup K) = \overline{M}^P_{\sum_{i=1}^s \Re_i}(G) \cup \overline{M}^P_{\sum_{i=1}^s \Re_i}(K),$

$$(PL_6) G \subseteq K \Rightarrow \underline{M}^P_{\sum_{i=1}^s \Re_i}(G) \subseteq \underline{M}^P_{\sum_{i=1}^s \Re_i}(K),$$

$$(PU_6) G \subseteq K \Rightarrow \overline{M}^P_{\sum_{i=1}^s \Re_i}(G) \subseteq \overline{M}^P_{\sum_{i=1}^s \Re_i}(K),$$

$$(PL_7)\underline{M}^P_{\sum_{i=1}^s \Re_i}(G \cup K) \supseteq \underline{M}^P_{\sum_{i=1}^s \Re_i}(G) \cup \underline{M}^P_{\sum_{i=1}^s \Re_i}(K),$$

$$(PU_7)\overline{M}^P_{\sum_{i=1}^s \Re_i}(G \cap K) \subseteq \overline{M}^P_{\sum_{i=1}^s \Re_i}(G) \cap \overline{M}^P_{\sum_{i=1}^s \Re_i}(K).$$

Example 3.6 (Continue on Example 3.3). Calculating the pessimistic MGRS-IVHFS $L\&U$ approximations of G as following steps:

$$h_{\underline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_1) = \{[0, 0.3]\},\ h_{\underline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_2) = \{[0.1, 0.3]\},$$

$$h_{\underline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_3) = \{[0.3, 0.5], [0.1, 0.4]\},\ h_{\underline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_4) = \{[0.1, 0.3], [0.1, 0.3]\}.$$

It can be computed that the pessimistic MGRS-IVHFS lower approximation of G is

$\underline{M}^P_{\sum_{i=1}^s \Re_i}(G) = \{\langle \tau_1, \{[0, 0.3]\}\rangle, \langle \tau_2, \{[0.1, 0.3]\}\rangle, \langle \tau_3, \{[0.3, 0.5], [0.1, 0.4]\}\rangle, \langle \tau_4, \{[0.1, 0.3]\}\rangle | \tau \in \mho\}.$

In the same manner, we are able to gain that

$$h_{\overline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_1) = \{[0.5, 0.7], [0.6, 0.9]\},\ h_{\overline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_2) = \{[0.5, 0.6], [0.5, 0.9]\},$$

$$h_{\overline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_3) = \{[0.6, 0.8]\},\ h_{\overline{M}^P_{\sum_{i=1}^s \Re_i}(G)}(\tau_4) = \{[0.3, 0.6], [0.6, 0.9]\}.$$

As a consequence, the pessimistic MGRS-IVHFS upper approximation of G is $\overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) = \{\langle\tau_1, \{[0.5, 0.7], [0.6, 0.9]\}\rangle, \langle\tau_2, \{[0.5, 0.6], [0.5, 0.9]\}\rangle, \langle\tau_3, \{[0.6, 0.8]\}\rangle,$ $\langle\tau_4, [0.3, 0.6], [0.6, 0.9]\}\rangle | \tau \in \mho\}$.

3.3. **The relationship between single-granulation and multi-granulation interval-valued hesitant fuzzy rough set.** With reference to preceding sections, we have brought forward some requisite concepts and relevent properties of the optimistic and pessimistic MGRS-IVHFS. Afterwards, we will study the relationship between IVHFRS, optimistic and pessimistic MGRS-IVHFS.

Proposition 3.7. *Suppose that $\Re_i \in \mathbb{IVHFR}(\mho\times\mho)$ $(i = 1, 2, \ldots, s)$ are s relations over the domain $\mho$. Given any $G \in IVHFS$, then*

$$(O_1)\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{\bigcup_{i=1}^{s}\Re_i}(G), \qquad \overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \supseteq \overline{\bigcup_{i=1}^{s}\Re_i}(G).$$

$$(O_2)\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) = \bigcup_{i=1}^{s}\underline{\Re_i}(G), \qquad \overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) = \bigcap_{i=1}^{s}\overline{\Re_i}(G).$$

$$(O_3)\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G\cap K) = \bigcup_{i=1}^{s}(\underline{\Re_i}(G)\cap\underline{\Re_i}(K)), \ \overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G\cup K) = \bigcap_{i=1}^{s}(\overline{\Re_i}(G)\cup\overline{\Re_i}(K)).$$

Proof. On account of the quantity of granulations is generally finite, the proof is only given under the circumstance of two relations for the convenience of description, that is $s = 2$. Suppose that interval-valued hesitant fuzzy relations are $\Re_i$ and $\Re_j$, respectively. Analogously, we certify merely the properties of lower approximation here.

(O_1) For every $\omega \in \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)$, it can be found from the Definition 3.1 that $h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\omega) = \bigvee_{i=1}^{s}\wedge_{\psi\in\mho}\{h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi)\} = \{\wedge_{\psi\in\mho}(h_{\Re_j^c}(\omega,\psi) \vee h_G(\psi))\} \vee \{\wedge_{\psi\in\mho}(h_{\Re_k^c}(\omega,\psi)\vee h_G(\psi))\}$. At the same time, for every $\omega \in \underline{\bigcup_{i=1}^{s}\Re_i}(G)$, $h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\omega)$ $= \wedge_{\psi\in\mho}\{h_{\bigcup_{i=1}^{s}\Re_i^c}(\omega,\psi)\vee h_G(\psi)\} = \wedge_{\psi\in\mho}\{h_{\Re_j^c\cup\Re_k^c}(\omega,\psi)\vee h_G(\psi)\} = \wedge_{\psi\in\mho}\{h_{\Re_j^c}(\omega,\psi)\vee h_{\Re_k^c}(\omega,\psi)\vee h_G(\psi)\}$. Since $h_{\Re_j^c}(\omega,\psi)\vee h_{\Re_k^c}(\omega,\psi)\succeq h_{\Re_j^c}(\omega,\psi)$ and $h_{\Re_j^c}(\omega,\psi)\vee h_{\Re_k^c}(\omega,\psi)$ $\succeq h_{\Re_k^c}(\omega,\psi)$, it is clear that $\{\wedge_{\psi\in\mho}(h_{\Re_j^c}(\omega,\psi)\vee h_G(\psi))\}\vee\{\wedge_{\psi\in\mho}(h_{\Re_k^c}(\omega,\psi)\vee h_G(\psi))\}$ $\leq \wedge_{\psi\in\mho}\{h_{\Re_j^c}(\omega,\psi) \vee h_{\Re_k^c}(\omega,\psi) \vee h_G(\psi)\}$, namely $h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\omega)\preceq h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\omega)$. So we have $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{\bigcup_{i=1}^{s}\Re_i}(G)$.

(O_2) For every $\omega \in \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)$,we can obtain from the Definition 3.1 that $h_{\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G)}(\omega) = \bigvee_{i=1}^{s} \wedge_{\psi\in\mathfrak{U}}\{h_{\Re_i^c}(\omega,\psi) \vee h_G(\psi)\} = \bigvee_{i=1}^{s} h_{\underline{\Re_i}(G)}(\omega)$. Therefore, $\underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) = \bigcup_{i=1}^{s} \underline{\Re_i}(G)$.

(O_3) It is toilless for us to prove these propositions directly by applying the result of Proposition 3.5 and (O_2). □

Proposition 3.8. *Suppose that $\Re_i \in \mathbb{IVHFR}(\mathfrak{U}\times\mathfrak{U})$ $(i = 1, 2, \ldots, s)$ are s relations over the domain $\mathfrak{U}$. Given any $G \in IVHFS$, then*

$$(P_1)\underline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{\bigcup_{i=1}^{s}\Re_i}(G), \qquad \overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) \supseteq \overline{\bigcup_{i=1}^{s}\Re_i}(G).$$

$$(P_2)\underline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) = \bigcap_{i=1}^{s}\underline{\Re_i}(G), \qquad \overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) = \bigcup_{i=1}^{s}\overline{\Re_i}(G).$$

$$(P_3)\underline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G\cap K) = \bigcap_{i=1}^{s}(\underline{\Re_i}(G)\cap\underline{\Re_i}(K)),\ \overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G\cup K) = \bigcup_{i=1}^{s}(\overline{\Re_i}(G)\cup\overline{\Re_i}(K)).$$

Proof. We can prove above properties readily by adopting similar approaches of Proposition 3.7. □

Proposition 3.9. *Suppose that $\Re_i \in \mathbb{IVHFR}(\mathfrak{U}\times\mathfrak{U})$ $(i = 1, 2, \ldots, s)$ are s relations over the domain $\mathfrak{U}$. Given any $G \in IVHFS$, then*

$$(M_1)\underline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{\bigcup_{i=1}^{s}\Re_i}(G), \quad \overline{\bigcup_{i=1}^{s}\Re_i}(G) \subseteq \overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G).$$

$$(M_2)\underline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \underline{\Re_i}(G) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G), \qquad \overline{M}^{O}_{\sum_{i=1}^{s}\Re_i}(G) \subseteq \overline{\Re_i}(G) \subseteq \overline{M}^{P}_{\sum_{i=1}^{s}\Re_i}(G).$$

Proof. (M_1) Applying the result of Definition 3.1 and 3.4, Proposition 3.7 and 3.8, it is easy to prove.

(M_2) Similarly, we can prove without effort from Proposition 3.7, 3.8 and (M_1). □

Example 3.10 (Continue on Example 3.3, 3.6). As is displayed below, we work out $\underline{\bigcup_{i=1}^{s}\Re_i}(G)$ and $\overline{\bigcup_{i=1}^{s}\Re_i}(G)$:

$$h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\tau_1) = \{[0, 0.3]\},\ h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\tau_2) = \{[0.2, 0.5], [0.3, 0.6]\},$$

$$h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\tau_3) = \{[0.6, 0.8]\},\ h_{\underline{\bigcup_{i=1}^{s}\Re_i(G)}}(\tau_4) = \{[0.1, 0.3], [0.4, 0.7]\}.$$

It follows that $\underline{\bigcup_{i=1}^{s} \Re_i(G)} = \{\langle \tau_1, \{[0, 0.3]\}\rangle, \langle \tau_2, \{[0.2, 0.5], [0.3, 0.6]\}\rangle, \langle \tau_3, \{[0.6, 0.8]\}\rangle, \langle \tau_4, [0.1, 0.3], [0.4, 0.7]\}\rangle | \tau \in \mathfrak{U}\}$.

In the same way, we can figure out $\overline{\bigcup_{i=1}^{s} \Re_i(G)} = \{\langle \tau_1, \{[0.2, 0.3], [0.3, 0.5]\}\rangle, \langle \tau_2, \{[0.2, 0.5], [0.3, 0.6]\}\rangle, \langle \tau_3, \{[0.6, 0.8]\}\rangle, \langle \tau_4, [0.1, 0.3], [0.4, 0.9]\}\rangle | \tau \in \mathfrak{U}\}$.

Eventually, it is obvious that

$$\underline{M}^{P}_{\sum_{i=1}^{s} \Re_i}(G) \subseteq \underline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G) \subseteq \underline{\bigcup_{i=1}^{s} \Re_i(G)} \subseteq G \subseteq \overline{\bigcup_{i=1}^{s} \Re_i(G)} \subseteq \overline{M}^{O}_{\sum_{i=1}^{s} \Re_i}(G) \subseteq \overline{M}^{P}_{\sum_{i=1}^{s} \Re_i}(G).$$

4. Conclusions

RS and FS are significant tools to dispose of the uncertainty. Plenty of relevant models can be acquired by extending the two theories. It is essential to select suitable models to deal with a variety of problems. In the light of the IVFS, we have come up with the MGRS-IVHFS theory, and studied the concepts and theorems of MGRS-IVHFS in the context of optimism and pessimism. What is more, we have demonstrated the validity of these theorems through several examples. By establishing a creative model that is involved in foregoing contents, we have extended the theory of IVHFRS from single granularity to multiple granularity, which is convenient for us to evaluate the decision-making process from multiple perspectives. Through several examples, we closely unify our model with realistic issues, but prospective research will not be confined to this sort of problem. As a matter of fact, this approach is beneficial to fit the era of big data nowadays. Whereas, what is noteworthy is that we cannot lose sight of these shortcomings concerning this model. For one thing, we have not set forth concrete criteria to sequence membership degrees between various objects. For another, when certain object transforms owing to an insertion or a deletion, its approximations need to be recalculated, which invests abundant time and vigor.

In the further research, we will extend this model to the information system. Aimed at attribute reduction, we intend to combine the model with machine learning method. Simultaneously, in allusion to the restrictions of our model, we are calculated to update approximations with dynamic data sets in the system.

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Manuscript received February 15, 2021
revised March 26, 2021

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